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FORMATION OF A SNOW-FIRN LAYER IN SURFACE MELTING OF SNOW

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We consider the problem of formation of a snow-firn stratum in a mass of snow. We assume that initially on the surface melt water is formed, which subsequently seeps through a porous medium and crystallizes due to cooling. The change in the porosity during the crystallization of the water in the mass of snow is analyzed.

Prediction of the laws governing the drainage of melt water from the surface of a mass of snow is an important problem in the hydrology of glaciers [1]. This is dictated by the need to predict the water volume of mountain rivers during the spring thawing of glaciers and to evaluate the stability of a snow slope and the possibility of formation and descent of snow avalanches.

One of the first models of a snow-firn stratum is the Colbeck model [2–4], in which basic approximations were formulated to calculate the flow of melt water in a mass of snow. In [5, 6], based on this model, more simplified approaches to integral evaluation of water flows were suggested. Further development of the model of evolution of a snow-firn stratum was presented in [7]; however, application of this model to the solution of practical problems involves difficulties in obtaining the information that constitutes the input data. In our opinion, the development of models that would take account of the main physical regularities of the process using measurable data is more promising. This approach to calculating a flow of water through a mass of snow was suggested for the first time in [8]. The present work is an extension of [8] to the case of account taken of the crystallization of water in the mass of snow of moderate glaciers whose temperature is close to that of the crystallization.

Physicomathematical Model and Formulation of the Problem. We assume that thawing of snow occurs on the surface of a mass of snow under the action of a heat flux, and a water film is formed. Formation of water on the surface of snow as a result of rainfall is also possible. Subsequently, water seeps into the mass of snow. This flow occurs under gravitational forces, and this leads to vertical filtration of water. The snow is a porous medium that has a lamellar structure. Its lamellar nature may exert a marked effect on the motion of water through the snow cover. We consider the snow cover to be a two-phase system consisting of ice and water. In flowing, water envelops particles of the medium, and as a result of crystallization, pores of the mass of snow fill up with ice crust.

"Ice interlayers" can appear in the mass of ice, whose formation is explained by crystallization of water when its temperature decreases in the process of filtration. It is taken into consideration below that crystallization of water causes a change in the porosity of the mass of snow, and this influences its speed of motion.

We assume that the motion of water in a mass of snow obeys the Darcy law. This law accociates the capillary-pressure gradient $\partial P_c / \partial z$ and the free-fall acceleration g with the volume of water u that flows through unit cross section in unit time:

$$u = \frac{k_w}{\mu_w} \left[\frac{\partial P_c}{\partial z} + \rho g \right] . \tag{1}$$

In the case of normal conditions of gravitational drainage in the snow, when flow occurs with gravitation taken into account but without capillary pressure, the Darcy law can be written in the form

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$$u = \frac{\rho k_w g}{\mu_w}. \tag{2}$$

The coefficient of permeability of water is decisive in the theory of two-phase flow with application of the Darcy law. There are many experimental and theoretical relationships between k_w and the saturation S_w (ratio of water and pore volumes). However, in the present work for subsequent calculations the notion of effective saturation with water S^* is introduced, which is understood to be the fraction of the volume of the pores that contains the liquid phase of water:

$$S^* \equiv \frac{S_w - S_p}{1 - S_p}. \tag{3}$$

Here, the coefficient of permeability k_w depends only on S^*. Morel-Seytoux [9] showed that $k_w \propto S^{*\sigma}$, where σ varies with the material, but for snow it is taken to be equal to three. Thus

$$k_w = kS^{*3}. \tag{4}$$

Using this relation, for vertical water flow we obtain

$$u = AkS^{*3}, \quad A = \frac{\rho g}{\mu_w}. \tag{5}$$

The basic equation of moisture transfer can be found from the continuity equation for water:

$$\frac{\partial \alpha S^*}{\partial t} + \frac{\partial u}{\partial z} = \frac{1}{\rho} \varepsilon_{12}, \quad \varepsilon_{12} = \rho_1 \frac{d\alpha}{dt}. \tag{6}$$

A change in the temperature in filtration of water in the mass of snow occurs because of convection, heat conduction, and heat exchange with the medium. It is described by the equation

$$C\rho \frac{\partial \alpha T}{\partial t} + C\rho \frac{\partial \alpha T u}{\partial z} - \lambda \frac{\partial}{\partial z} \alpha \frac{\partial T}{\partial z} + h_m (T - T_m) - L_{12}\varepsilon_{12} = 0. \tag{7}$$

We assume that at the initial instant of time the change in the temperature of the medium can be neglected, since the mass of melt water is much smaller than the mass of snow:

$$T(z, 0) = T_m(z). \tag{8}$$

It is assumed that the temperature of the water on the surface of the mass of snow differs insignificantly from the phase change temperature:

$$T(0, t) = \varphi(t). \tag{9}$$

The heat flux at the leading edge of the moving water is assumed to be equal to

$$q(H, t) = \lambda \left. \frac{dT}{dz} \right|_{z=H} = 0.05 \ \mathrm{W/m^2}. \tag{10}$$

Substituting u from (5) into (6), we obtain an equation for the saturation:

$$\alpha \frac{\partial S^*}{\partial t} + 3AkS^{*2} \frac{\partial S^*}{\partial z} = 0. \tag{11}$$

The mathematical model for the change in the saturation of the mass of snow with water is described by Eq. (11) subject to initial and boundary conditions in the form

$$S^* (z = 0, t) = S_0 (t) , \; S^* (z, t = 0) = F (z) .$$

In solving the problem of the distribution of the temperature during the flow of water two cases are possible.

1. The temperature of the water does not coincide with the temperature of its crystallization at the specified depth. Here, the problem of the heat conduction of the water

$$C\rho \frac{\partial \alpha T}{\partial t} + C\rho \frac{\partial \alpha T u}{\partial z} - \lambda \frac{\partial}{\partial z} \alpha \frac{\partial T}{\partial z} + h_m (T - T_m) = 0 \tag{12}$$

under the specified boundary and initial conditions (8)-(10) is solved without account for the phase transition: $\varepsilon_{12} = 0$. The porosity of the mass of snow remains constant.

2. The temperature of the water is compared with the temperature of crystallization of ice at the specified depth. Crystallization of water and its conversion into ice occur. The amount of ice formed is found by solving the plane problem of phase transition on the surface of pores. Since solidification of the mass of water $\rho \Delta \xi$ is accompanied by liberation of the quantity of heat $L_{12} \rho \Delta \xi$, the energy balance leads to the equation

$$\lambda_1 \frac{\partial T_m}{\partial x} - \lambda \frac{\partial T}{\partial x} = L_{12} \rho \frac{d\xi}{dt} . \tag{13}$$

Assuming that a temperature $T_m(z) < 0$ is maintained on the surface of a crystal $x = 0$ and the phase change temperature is equal to zero, we see that the boundary of freezing $x = \xi$ moves with time to the side of the liquid. With the above initial conditions for the medium and the phase transition Eq. (13) can be represented in the form [10]

$$\frac{1}{\sqrt{\pi}} \frac{\exp (-\beta^2)}{\Phi(\beta)} = - D\beta , \tag{14}$$

where

$$D = \frac{L_{12} \rho \lambda_1}{C_1^2 \rho_1^2 T_m} < 0 .$$

Equation (14) yields the value of β at each point z.

The change in the porosity of the mass of snow occurs according to the formula

$$\alpha = \alpha_0 - 2B \frac{\lambda \beta}{C\rho r} \sqrt{t} , \tag{15}$$

obtained on the assumption of growth of each individual crystal in unit volume. The value of the constant B depends on the geometry of the pore space. For crystals of spherical shape it was taken to be equal to $\pi/2$. The value of this quantity was varied in carrying out calculations.

The problem of formation of the structure of a snow-firn layer in surface thawing of snow was solved numerically. On each time layer, first the temperature field (11) was determined with conditions (8)-(10), then the temperature of the water and the temperature of crystallization were compared at each point, and at the points where these temperatures coincided, transition to formula (14) was made. As a result, at the points where the temperature of the water coincided with the crystallization temperature, the amount of water that had passed into the crystalline state was determined. Next, the saturation was found from Eq. (11). Then, the velocity of water flow was determined from formula (5), and transition to the next time layer was made.

The dependence of the quantity of ice on the depth can be represented by the following formula:

$$m (z, t) = \rho_1 (\alpha (z, t) - \alpha_0) . \tag{16}$$

The total quantity of ice formed in the mass of snow can be evaluated by an integral in the form

$$M(t) = \int_0^H m(z, t)\, dz\,. \tag{17}$$

The solution of the above problem allows us to calculate the quantity of ice at each point of the mass of snow.

Method of Solution. The solution of the problem of the extent of saturation of a mass of snow with water can be obtained analytically by the method of characteristics. Assuming the coefficient of permeability to be constant, we have an equation for the characteristic in the form

$$\left.\frac{dz}{dt}\right|_{S^*} = 3\,\frac{Ak}{\alpha}\,S^{*2}\,. \tag{18}$$

The solution of this equation yields

$$z = 3\,\frac{Ak}{\alpha}\,F(z_0)^2\,t + z_0\,, \tag{19}$$

where $F(z_0)$ is the initial value for saturation. Solving Eq. (19) for z_0, we obtain $z_0 = \Psi(z)$. Substitution of this value into the initial function for saturation allows us to obtain

$$S^* = F(z_0) = F(\Psi(z))\,. \tag{20}$$

The problem of the distribution of the temperature field in filtration of water in a mass of snow was solved by a numerical method. For this purpose, in Eq. (12) and in the corresponding initial and boundary conditions the following dimensionless parameters are used:

$$z = Lz'\,,\quad t = \frac{C\rho L^2}{\lambda}\,t'\,,\quad u = \frac{\lambda}{C\rho L}\,u'\,,\quad h_{\mathrm{m}} = \frac{\lambda}{L^2}\,h_{\mathrm{m}}' \tag{21}$$

(below, the primes are dropped). The heat conduction equation (12) can be written here in the form

$$\frac{\partial \alpha T}{\partial t} + \frac{\partial \alpha T u}{\partial z} - \frac{\partial}{\partial z}\,\alpha\,\frac{\partial T}{\partial z} + h_{\mathrm{m}}(T - T_{\mathrm{m}}) = 0\,. \tag{22}$$

The differential operators in this equation were replaced by difference ones, and, as a result, to solve the problem of the temperature field, the following difference scheme was obtained:

$$\begin{aligned}
&T_{j-1}^{n+1}\left[\frac{\tau}{h}\,\alpha_{j-1}^{n+1}\,u_{j-1}^{n+1} + \frac{\tau}{h^2}\,\frac{\alpha_j^{n+1} + \alpha_{j-1}^{n+1}}{2}\right] - \\
&- T_j^{n+1}\left[\alpha_j^{n+1} + \frac{\tau}{h}\,\alpha_j^{n+1}\,u_j^{n+1} + \frac{\tau}{h^2}\,\frac{\alpha_j^{n+1} + 2\alpha_j^{n+1} + \alpha_{j-1}^{n+1}}{2} + h_{\mathrm{m}}\tau\right] + \\
&+ T_{j+1}^{n+1}\left[\frac{\tau}{h^2}\,\frac{\alpha_{j+1}^{n+1} + \alpha_j^{n+1}}{2}\right] = -\,\alpha_j^n\,T_j^n - \tau\,h_{\mathrm{m}} T_{\mathrm{m}j}\,,
\end{aligned} \tag{23}$$

$$j = 1, 2, \ldots, J - 1\,;\quad n = 0, 1, \ldots, N - 1\,.$$

The initial and boundary conditions that take account of the second-order approximation, with regard for Eqs. (8)-(10), were taken in the form

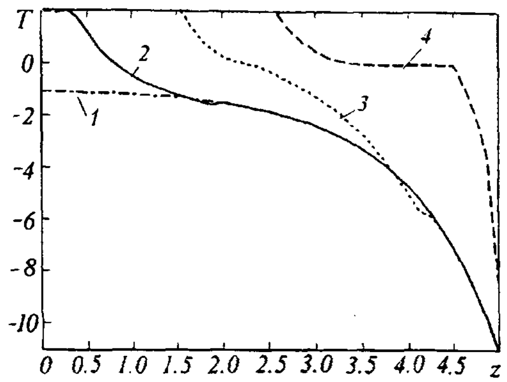

Fig. 1. Temperature of water vs. depth at different instants of time: 1) at the initial instant of time; 2) 1 h; 3) 5 h; 4) 24 h after the start of thawing. T, $^{\circ}$C; z, m.

$$T_0^n = \varphi(\tau n\chi)\,,\ \ \chi = \frac{C\rho L^2}{\lambda}\,,\ \ n = 0, 1, \ldots, N\,; \tag{24}$$

$$T_{J-1}^{n+1} - T_J^{n+1}\left[1 + \frac{h^2}{2\tau} + \frac{h^2}{2\tau}\left(\frac{u_J^{n+1} - u_{J-1}^{n+1}}{h} + h_{\mathrm{m}}\right)\right] = -\frac{hq(H, \tau n\chi)}{\lambda} -$$

$$-\frac{h^2}{2\tau}\frac{\alpha_J^n}{\alpha_J^{n+1}}\varphi(\tau n\chi) + \frac{h^2}{2\alpha_J^{n+1}}\left[u_J^{n+1}\frac{q(H, \tau n\chi)}{\lambda} - \frac{\alpha_J^{n+1} - \alpha_{J-1}^{n+1}\, q(H, \tau n\chi)}{\lambda} - h_{\mathrm{m}} T_{\mathrm{m}J}\right], \tag{25}$$

$$n = 0, 1, \ldots, N - 1\,;$$

$$T_j^0 = T_{\mathrm{m}}(jhL)\,,\ \ j = 0, 1, \ldots, J\,. \tag{26}$$

The system of algebraic equations with a tridiagonal matrix (17)-(20) was solved on each time layer by the pivotal method.

Results of Numerical Simulation. The algorithm written above to calculate the saturation with water and the dependence of the water temperature on the depth was used to evaluate the formation of the structure of a snow-firn layer in surface thawing of snow.

The initial profile of the temperature of the mass of snow over its depth was selected according to experimental data of [2], which can be approximated by the following relation:

$$T_{\mathrm{m}}(z) = -10\exp\left(\frac{z - H}{H}\right) - 1\,.$$

In the present work we studied the influence of a heat flux incident on the surface of snow on the formation of a snow-firn stratum. The results of the numerical simulation showed that the porosity changes slightly over the depth with a change in the heat flux. This seems to be explainable by the fact that the dependence of the saturation with water on the time on the surface of the mass of snow exerts a decisive effect on the formation of a snow-firn stratum. When $S^*(z = 0, t) = 1$, the porosity of the medium changes most intensely.

Results of calculation of the temperature field in filtration of water in a mass of snow at different instants of time are presented in Fig. 1. It is seen from the figure that with time the heat wave propagates together with the water flow.

Figure 2 illustrates the dynamics of the change in the porosity of the mass of snow when the saturation of its surface with water is equal to 0.5. Figure 2a corresponds to the case where the initial permeability and porosity decrease with the depth [3]:

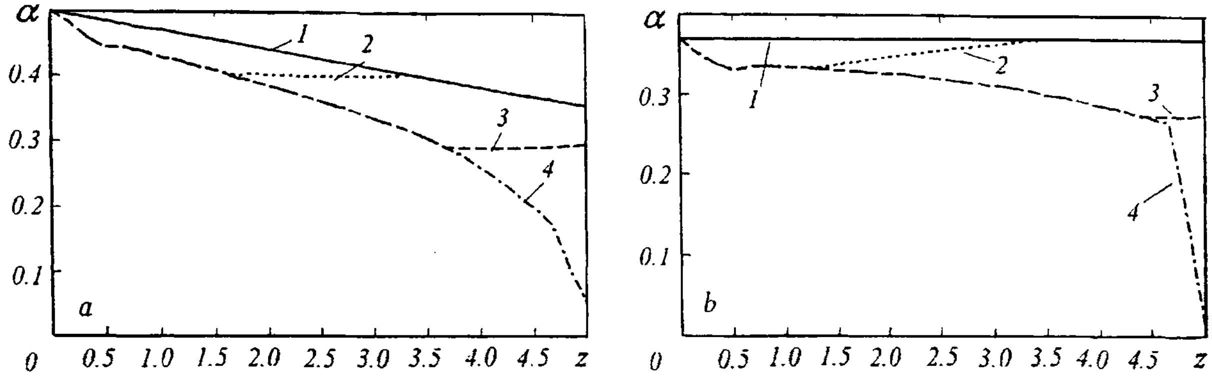

Fig. 2. Values of the porosity of the mass of snow at different instants of time [a) for an initial porosity and permeability that depend on the depth; b) for constant initial porosity and permeability]: 1) at the initial instant of time; 2) 2 h; 3) 8 h; 4) 80 h after the start of thawing.

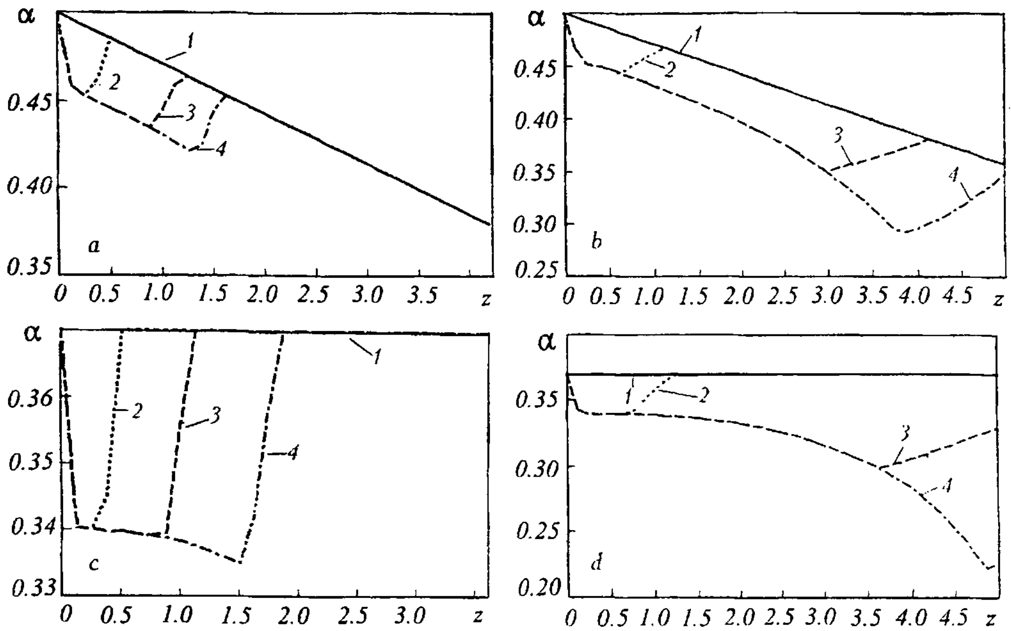

Fig. 3. Porosity of the mass of snow vs. depth for different values of the effective saturation on the surface [a, c) 0.1; b, d) 0.3] and for an initial permeability and porosity that vary with the depth (a, b) and constant initial permeability and porosity (c, d): 1) at the initial instant of time; 2) 1 h; 3) 5 h; 4) 24 h after the start of thawing.

$$k = c \exp (b\alpha), \quad c = 0.65 \cdot 10^{-13} \ \mathrm{m}^2, \quad b = 15.9, \quad \alpha_0 = d_1 + d_2 z,$$

$$d_1 = 0.5, \quad d_2 = -0.287 \cdot 10^{-1} \ \mathrm{m}^{-1}.$$

Figure 2b corresponds to constant initial permeability and porosity of the mass of snow. It follows from the results of calculation that, the surface layer being fully saturated with water, formation of an ice interlayer occurs 80 h after the start of the process of water filtration. By this time the porosity of the snow has attained zero value, and penetration of water into deeper layers of the mass of snow becomes impossible. After this, one should expect filling of the upper portion of the porous mass of snow with water, as a result of which the density of the wet snow is increased considerably. It is the weighting of the upper portion of the snow cover that leads to the loss of stability of snow avalanches located on mountain slopes. In actual fact, the variants of calculations for full saturation of the

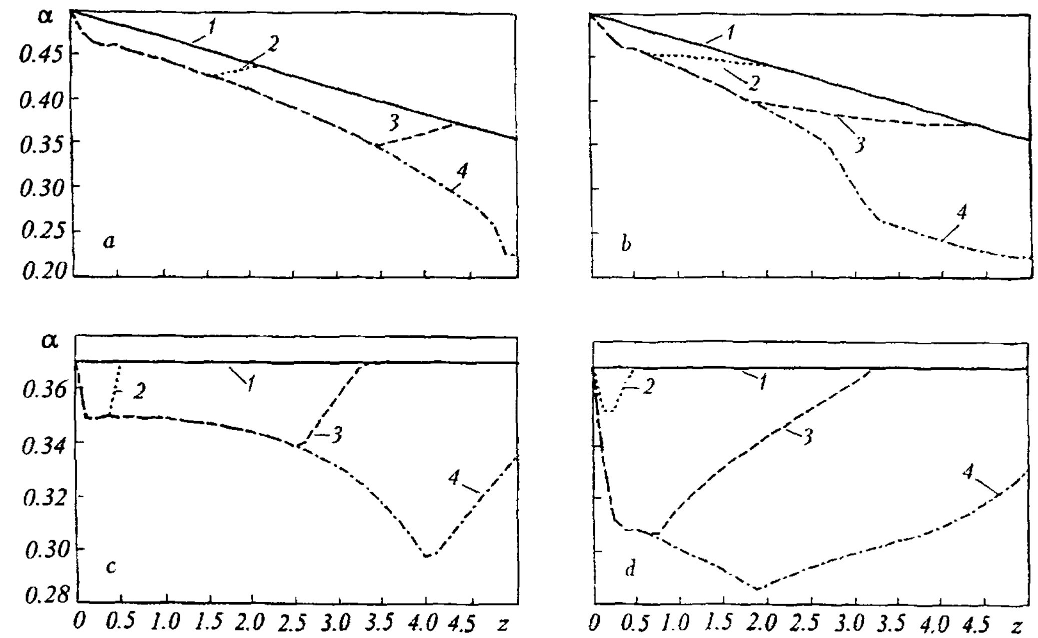

Fig. 4. Porosity of the mass of snow vs. depth for different values of the heat capacity: for an initial porosity and permeability that depend on the depth (a and b) and for constant initial porosity and permeability (c and d) [a, c) C = 34.71 J/(mole·K); b, d) 123.15]; 1-4) notation as in Fig. 3.

surface layer with water are model ones, and usually they do not correspond to the actual processes. The observed values of saturation in surface thawing of a mass of snow are usually evaluated as hundredths [3].

Figure 3a and b presents data on the effect of saturation of the snow surface with water on the change in the snow-firn layer when the initial permeability and porosity change over the depth. As expected, the value of the effective saturation with water $S^*(z = 0, t)$ on the surface of the snow substantially determines the evolution of the process. With a decrease in the saturation, the region of formation of ice in the mass of snow is wider, and the process is slower.

The same processes at constant initial porosity and permeability are shown in Fig. 3c and d. The main specific features of the evolution of the process in these cases are the same as those discussed above for variable porosity and permeability.

Figure 4 demonstrates the effect of the heat capacity on the dynamics of formation of a snow-firn layer when the initial porosity and permeability are constant (Fig. 4a and b) and when the permeability and porosity depend on the depth (Fig. 4c and d). It is seen from the figures that an increase in the value of the heat capacity leads to a more rapid decrease in the porosity of the mass of snow. This is explained by the fact that the main mechanism of formation of ice in the mass of snow is caused by filtration of water in the porous medium, which leads to a much greater crystallization of water on increase in its heat capacity.

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NOTATION

u, rate of water filtration; k_w, water permeability; P_c, capillary pressure; g, free-fall acceleration; k, intrinsic permeability of snow in the unsaturated zone; μ_w, water viscosity; ρ, density of water; ρ_1, density of ice; z, distance from the surface of the mass of snow; t, time; S^*, effective saturation; S_w, saturation; S_p, permanent saturation; α, porosity of the mass of snow; α_0, distribution of the porosity of the mass of snow over the depth at the initial instant of time; ε_{12}, mass of water that converted into ice in unit time in unit isolated volume; L_{12}, specific heat of transition from water into ice; λ, coefficient of thermal conductivity of water; λ_1, coefficient of thermal conductivity of ice; C, coefficient of specific heat of water; C_1, coefficient of specific heat of ice; T, temperature of water; L, characteristic dimension of length used in nondimensionalization; h_m, coefficient of heat exchange with the medium; H, depth of the mass of snow; $T_m(z)$, temperature of the medium; $\varphi(t)$, temperature of water on the surface of the mass of snow; $S_0(t)$, saturation maintained on the surface of the mass of flow; $F(z)$, distribution of saturation over the depth at the initial time instant; ξ, boundary of freezing; x, distance from the surface of a crystal; $m(z, t)$, mass of ice formed; $M(t)$, total quantity of ice formed in the mass of snow; d_1, value of the porosity on the surface at the initial instant of time; r, radius of a crystal in the mass of snow; T_j^n grid function for the temperature of water; u_j^n, grid function for the rate of filtration; α_j^n, grid function for the porosity; h, step in the coordinate; τ, step in time; $\Phi(\beta)$, error integral; B, parameter that depends on the geometry of the pore space; b, c, constants in the formula for the permeability; d_2, constant in the formula for the initial porosity; D, dimensional parameter in Eq. (14); χ, dimensional parameter in formula (24); σ, exponent equal to 3.

REFERENCES

1. J. Hardy, M. Albert, and P. Marsh, *Abstracts of Int. Conf. "Snow Hydrology. The Integration of Physical, Chemical and Biological Systems,"* Brownsville, Vermont, CRREL Report 98-10, Hanover (1998).
2. S. C. Colbeck, *J. Glaciology,* **26**, No. 94, 291-301 (1980).
3. S. C. Colbeck, *J. Glaciology*, **11**, No. 63, 369-396 (1972).
4. S. C. Colbeck, *J. Glaciology*, **13**, No. 67, 85-97 (1974).
5. J. Martinec and A. Rango, *The Snowmelt Runoff Model (SRM) User's Manual*, NASA Reference Publ. 1100, NASA, Washington, DC (1983).
6. W. Kustas, A. Rango, and R. Uijenhoet, *Water Resource Research*, **30**, 1515-1527 (1994).
7. R. Jordan, *A One-Dimensional Temperature Model for a Snow Cover,* Special Report 91-16, US Army Corps of Engineers, CRREL, Hanover, NH (1991).
8. M. Albert and G. Krajesky, *Hydrological Processes*, **12**, 1809-1824 (1998).
9. H. J. Morel-Seytoux, (in: R. J. M. De Wiest (ed.), *Flow through Porous Media*, New York (1969), pp. 455-516.
10. A. N. Tikhonov and A. A. Samarskii, *Equations of Mathematical Physics* [in Russian], Moscow (1966).